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Embedding on alphabet overlap digraphs

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Abstract Alphabet overlap digraphs can be viewed as a generalization of directed de Bruijn graphs. Given three integers $\alpha \geq 1$, $k \geq 2$ and $1 \leq i < k$, the alphabet overlap digraph $O(\alpha, k; i)$ is a digraph: the set of all words of length k over a certain alphabet with cardinality α is vertex set, and there is an arc from a vertex u to a vertex v if and only if the word of last $k - i$ letters of u coincides with the word of first $k - i$ letters of v. In this paper, we consider whether $O(\alpha, k; i)$ can be embedded in $O(\alpha, k; j)$ for given integers $1 \leq i < j < k$. In order to resolve this problem, we give an $O(1)$-time algorithm to decide whether there exists a permutation on $\{1, \ldots, k\}$ from $O(\alpha, k; i)$ to $O(\alpha, k; j)$. If such a permutation exists, for any vertex of $O(\alpha, k; i)$, we apply the permutation to change its label's position and map it to a vertex of $O(\alpha, k; j)$. Furthermore, we obtain an embedding from $O(\alpha, k; i)$ to $O(\alpha, k; j)$. Hence, we solve partly the problem. As a consequence, we show that every directed de Bruijn graph can be embedded in all alphabet overlap digraphs with the same parameters α and k.

Keywords DNA graph · Directed de Bruijn graphs · Alphabet overlap digraphs · Embedding

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1 Introduction

For given two integers $\alpha \geq 1$ and $k \geq 2$, *directed de Bruijn graph* $B(\alpha, k)$ [3] is a directed graph with a^k vertices labeled by the words of length k over a certain alphabet with cardinality α: there is an arc from a vertex v labeled by $(v_1, \ldots, v_2)$ to a vertex w labeled by $(w_1, \ldots, w_k)$ if and only if $v_i = w_{i-1}$ for $i = 2, \ldots, k$. The out-degree and in-degree of each vertex are both equal to α. *Alphabet overlap digraph* $O(\alpha, k; i)$ $(1 \leq i < k)$ can be viewed as a generalization of $B(\alpha, k)$. $O(\alpha, k; i)$ is a directed graph with the same vertex set as that of $B(\alpha, k)$. Furthermore, there is an arc from a vertex v labeled by $(v_1, \ldots, v_2)$ to a vertex w labeled by $(w_1, \ldots, w_k)$ if and only if $v_j = w_{j-i}$ for $j = i + 1, \ldots, k$. The out-degree and in-degree of each vertex are both equal to α^i. We can see that $O(\alpha, k; 1) \cong B(\alpha, k)$ and the line digraph of $O(\alpha, k; i)$ is isomorphic to $O(\alpha, k + i; i)$.

Błażewicz et al. [2] introduced DNA graph which can be viewed as a vertex induced subgraph of directed de Bruijn graph $B(4, k)$ for some integer k with the alphabet $\{A, C, G, T\}$ corresponding to the four nucleotides of DNA chains: adenine, cytosine, guanine and thymine. Pendavingh et al. [6] showed that it is a NP-hard problem to decide whether a given digraph is a DNA graph. Wang et al. [7] generalized the definition of DNA graph as a vertex induced subgraph of alphabet overlap digraph $O(4, k; i)$ for some integers k and i with $1 \leq i < k$. Then, they showed that a digraph is a DNA graph if and only if it is a line digraph.

Recently, Godbole et al. [4] introduced alphabet overlap graph $G(\alpha, k; i)$, which can be viewed as the graph obtained from the underlying graph of $O(\alpha, k; i)$ by deleting all its loops. In [4], they showed that $G(\alpha, k; i)$ is Hamiltonian, and obtained that $\chi(G(\alpha, k; i)) = \alpha^{2i-k} + \alpha^{k-i}$ when $i \geq k/2$, and $\chi(G(\alpha, k; i)) \leq 1 + \alpha^{k-i}$ when $i < k/2$.

Now, we consider two digraphs $D_1 = O(\alpha, k; i)$ and $D_2 = O(\alpha, k; j)$ with $i < j$. Since $V(D_1) = V(D_2)$ and for any vertex v, $d_{D_1}(v) = \alpha^i < \alpha^j = d_{D_2}(v)$, we guess that D_1 is a spanning subgraph of D_2. But, this is not always true. For example, let $\alpha = 2, k = 4, i = 1$ and $j = 2$. We can see that $(2, 1, 2, 1)$ is an out-neighbor of $(1, 2, 1, 2)$ in D_1, but it is not an out-neighbor in D_2. So we pose another problem: is D_1 isomorphic to a spanning subgraph of D_2, i.e., can D_1 be embedded in D_2?

In this paper, we first show that D_1 can be embedded in D_2 when $i + j \geq k$. The main idea of this solution is to find a suitable permutation on $\{1, \ldots, k\}$. For any vertex of D_1, we can map it to a vertex of D_2 by applying the permutation to change its label's positions. Furthermore, this map gives us an embedding of D_1 into D_2. Hence, in Sect. 3, we consider whether such a permutation exists when $i + j < k$. When $2j \geq k$, we show that the answer is true if and only if an inequality holds. When $2j < k$, we give an $O(1)$-time algorithm to determine whether such a permutation exists. As an example, we consider the case $i = 1$ in Sect. 4, and show that such a permutation always exists for any integers j and k. Accordingly, we prove that very directed de Bruijn graph can be embedded in all alphabet overlap digraphs with the same parameters α and k. As corollaries of this result, all alphabet overlap digraphs are pancyclic and D_1 can be embedded into D_2 if i is a common factor of k and j. We conclude this paper with our main results (Theorem 5.1).

2 Case $i + j \geq k$

In the following, we suppose that $\alpha > 0$ and $k > 1$ are two integers. For convenience, we use D_1 and D_2 to denote $O(\alpha, k; i)$ and $O(\alpha, k; j)$, respectively, where i, j are two integers with $1 \leq i < j < k$. The main method to resolve the problem is to find a bijection f from $V(D_1)$ to $V(D_2)$ such that for any two vertices u and v in D_1, if (u, v) is an arc of D_1, then $(f(u), f(v))$ is also an arc of D_2.

Lemma 2.1 *Let α, k, i, j, D_1 and D_2 be defined as above. If $i + j \geq k$, then D_1 can be embedded in D_2.*

Proof We define a map f as follows: for any vertex $v = (l_1(v), \ldots, l_{j-i}(v), l_{j-i+1}(v), \ldots, l_j(v), l_{j+1}(v), \ldots, l_k(v))$ in D_1, let

$$\begin{aligned} f(v) &= (l_1(f(v)), \ldots, l_i(f(v)), l_{i+1}(f(v)), \ldots, l_j(f(v)), l_{j+1}(f(v)), \ldots, l_k(f(v))) \\ &= (l_{j-i+1}(v), \ldots, l_j(v), l_1(v), \ldots, l_{j-i}(v), l_{j+1}(v), \ldots, l_k(v)). \end{aligned}$$

We can see that for any vertex v, $f(v)$ is obtained from v by applying a fixed permutation to change the position of its label. So f is a bijection and we have

$$l_t(v) = \begin{cases} l_{t+i}(f(v)), & 1 \leq t \leq j - i, \\ 1_{t-j+i}(f(v)), & j - i + 1 \leq t \leq j, \\ l_t(f(v)), & j + 1 \leq t. \end{cases} \tag{2.1}$$

Then, we shall show that D_1 can be embedded in D_2 under bijection f. Let u and v be two vertices of D_1 such that (u, v) is an arc of D_1. By the definition of the alphabet overlap digraphs, we have $(l_{i+1}(u), \ldots, l_k(u)) = (l_1(v), \ldots, l_{k-i}(v))$. For any integer $t \geq j + 1 > i + 1$, by Eq. 2.1, we have

$$l_t(f(u)) = l_t(u) = l_{t-i}(v) = l_{t-i-j+i}(f(v)) = l_{t-j}(f(v)).$$

Hence, $(f(u), f(v))$ is an arc of D_2. □

3 Case $i + j < k$

In this section, we consider the case: $i + j < k$. In Sect. 2, we find a permutation on $\{1, \ldots, k\}$ from which we can deduce a bijection from $V(D_1)$ to $V(D_2)$. And from this bijection, we obtain an embedding from D_1 to D_2 when $i + j \geq k$. For convenience, this permutation is called a *permutation* from D_1 to D_2. Naturally, we propose another problem as follows. For any integers $1 \leq i < j < k$, is there a permutation Π on $\{1, \ldots, k\}$ from D_1 to D_2? Unfortunately, this problem is not always true. For example, there is no such a permutation from $O(2, 6; 2)$ to $O(2, 6; 3)$. Hence, the main work of this section is to decide whether such a permutation exists.

Let D_1 and D_2 be defined as above. Suppose that there is a permutation $\Pi = \binom{1\ 2\ \cdots\ k}{\pi_1\ \pi_2\ \cdots\ \pi_k}$ from D_1 to D_2. This implies that for any two vertices $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$ of D_1, if (u, v) is an arc of D_1, then $(f(u), f(v))$ is also an arc of D_2 where f is a bijection from $V(D_1)$ to $V(D_2)$ obtained from Π as Sect. 2. Since (u, v) is an arc of D_1, $v = (u_{1+i}, \ldots, u_{k+i})$. Then, $f(u) = (u_{\pi_1}, u_{\pi_2}, \ldots, u_{\pi_k})$ and $f(v) = (v_{\pi_1}, v_{\pi_2}, \ldots, v_{\pi_k}) = (u_{\pi_1+i}, u_{\pi_2+i}, \ldots, u_{\pi_k+i})$. Because $(f(u), f(v))$ is also an arc of D_2, we have that

$$(u_{\pi_{j+1}}, \ldots, u_{\pi_k}) = (v_{\pi_1}, \ldots, v_{\pi_{k-j}}) = (u_{\pi_1+i}, \ldots, u_{\pi_{k-j}+i}).$$

This implies the following system of equations

$$\pi_1 + i = \pi_{1+j}, \ldots, \pi_{k-j} + i = \pi_k. \tag{3.1}$$

Hence, we obtain that there exists a permutation Π from D_1 to D_2 if and only if system of Eq. 3.1 has an integer solution $(\pi_1, \ldots, \pi_k)$ such that $1 \leq \pi_i \leq k$ and distinct π_i have different values. We call such a solution a *feasible solution*.

In the following, we shall consider whether system of Eq. 3.1 has a feasible solution. For any two integers i and j with $i \leq j$, we use interval $[\ i, j\ ]$ to denote the integer set $\{i, i+1, \ldots, j\}$. Firstly, we classify the problem into two cases according to whether there are variables appear in the both sides of (3.1), i.e., whether $[\ 1, k-j\ ] \bigcap [\ j+1, k\ ] = \emptyset$.

Then, we consider the first case: $[\ 1, k-j\ ] \bigcap [\ j+1, k\ ] = \emptyset$, i.e., $k \leq 2j$. This implies that any variable in the left side does not appear in the right side of system of Eq. 3.1. Let n_1 and m_1 be two integers with $k = n_1 i + m_1 (1 \leq m_1 \leq i)$ and G a graph with vertex set $\{1, \ldots, k\}$. For any two vertices u and v of G, they are adjacent if and only if $|u - v| = i$. Clearly, G is composed of m_1 paths with length n_1 and $i - m_1$ paths with length $n_1 - 1$, i.e., $G \cong m_1 P_{n_1+1} \bigcup (i - m_1) P_{n_1}$, where P_{n_1} is a path with length $n_1 - 1$. We can see that (3.1) has a feasible solution if and only if G has a matching with cardinality $k - j$. Since $G \cong m_1 P_{n_1+1} \bigcup (i - m_1) P_{n_1}$, the cardinality of its maximum matching is $m_1 \lfloor \frac{n_1+1}{2} \rfloor + (i - m_1) \lfloor \frac{n_1}{2} \rfloor$. Accordingly, we obtain the following result.

Lemma 3.1 *Let α, k, i, j, n_1, m_1, D_1 and D_2 be defined as above. Suppose that $i + j < k$ and $2j \geq k$. There is a permutation from D_1 to D_2 if and only if $m_1 \lfloor \frac{n_1+1}{2} \rfloor + (i - m_1) \lfloor \frac{n_1}{2} \rfloor \geq k - j$. Furthermore, if $m_1 \lfloor \frac{n_1+1}{2} \rfloor + (i - m_1) \lfloor \frac{n_1}{2} \rfloor \geq k - j$, D_1 can be embedded in D_2.*

For example, let $D_1 = O(2, 6; 2)$ and $D_2 = O(2, 6; 3)$. Then $n_1 = 2$, $m_1 = 2$ and

$$m_1 \lfloor \frac{n_1+1}{2} \rfloor + (i - m_1) \lfloor \frac{n_1}{2} \rfloor = 2 < 3 = k - j.$$

Hence, there is no such a permutation from $O(2, 6; 2)$ to $O(2, 6; 3)$ as we mentioned at the beginning of this section.

Finally, we consider the other case: $[\ 1, k-j\ ]\bigcap[\ j+1, k\ ] \neq \emptyset$, i.e., $k > 2j$. Let n_2 and m_2 be two integers with $k = n_2 j + m_2 (1 \leq m_2 \leq j)$. In this case, (3.1) is equivalent to

$$\left\{\begin{array}{llll} \pi_1 + n_2 i & = \cdots = & \pi_{1+(n_2-1)j} + i & = \pi_{1+n_2 j} \\ \vdots & \vdots & \vdots & \vdots \\ \pi_{m_2} + n_2 i & = \cdots = & \pi_{m_2+(n_2-1)j} + i & = \pi_{m_2+n_2 j} \\ \pi_{m_2+1} + (n_2-1)i & = \cdots = & \pi_{m_2+1+(n_2-1)j} & \\ \vdots & \vdots & \vdots & \\ \pi_j + (n_2-1)i & = \cdots = & \pi_{j+(n_2-1)j} & \end{array}\right. \tag{3.2}$$

Let G be constructed as above case and $G^{'} = m_2 P_{n_2+1} \bigcup (j - m_2) P_{n_2}$. We can see that the system of Eq. 3.2 has a feasible solution if and only if $G^{'}$ is a spanning subgraph of G. This means that for every path of G, we can divide it into some paths with length n_2 or $n_2 - 1$, and we obtain exactly m_2 paths with length n_2 and $j - m_2$ paths with length $n_2 - 1$ in the whole graph G. Hence, (3.1) has a feasible solution if and only if the following system of equations

$$\left\{\begin{array}{lllll} \alpha_1 n_2 & + & \beta_1 (n_2+1) & = & n_1 + 1 \\ \vdots & & \vdots & & \vdots \\ \alpha_{m_1} n_2 & + & \beta_{m_1} (n_2+1) & = & n_1 + 1 \\ \alpha_{m_1+1} n_2 & + & \beta_{m_1+1} (n_2+1) & = & n_1 \\ \vdots & & \vdots & & \vdots \\ \alpha_i n_2 & + & \beta_i (n_2+1) & = & n_1 \end{array}\right. \tag{3.3}$$

has a non-negative integers solution $(\alpha_1, \ldots, \alpha_i, \beta_1, \ldots, \beta_i)$ satisfying the constraint condition

$$\sum_{t=1}^{i} \alpha_t = j - m_2, \quad \sum_{t=1}^{i} \beta_t = m_2. \tag{3.4}$$

Since the system of equations

$$\left\{\begin{array}{l} x n_2 + y(n_2+1) = k \\ x + y = j \end{array}\right.$$

has a unique solution $(x, y) = (j - m_2, m_2)$, the constraint condition (3.4) can be replaced by

$$\sum_{t=1}^{i} (\alpha_t + \beta_t) = j. \tag{3.5}$$

In the following, we shall resolve the system of Eq. 3.3 with constraint condition (3.5). Obviously, $-(n_1+1)n_2+(n_1+1)(n_2+1)=n_1+1$. Let $\alpha=-(n_1+1)$, $\beta=n_1+1$ and $q=\lceil\frac{\alpha}{n_2+1}\rceil$. Set $\alpha:=\alpha+q(n_2+1)$ and $\beta:=\beta-qn_2$. Then,

$$\begin{aligned}\alpha n_2+\beta(n_2+1)&=(-n_1-1+q(n_2+1))n_2+(n_1+1-qn_2)(n_2+1)\\&=-(n_1+1)n_2+(n_1+1)(n_2+1)=n_1+1\end{aligned}$$

and α is the minimum non-negative integer with $\alpha n_2+\beta(n_2+1)=n_1+1$. Now, if $\beta<0$, this implies that there does not exist non-negative integer solution of this equation. Similarly, we can decide whether equation $\alpha n_2+\beta(n_2+1)=n_1$ has a non-negative integer solution. Hence, we can determine whether (3.3) have non-negative integer solutions.

Then, we will consider the constraint condition (3.5). Let $A=(\alpha_1,\ldots,\alpha_i,\beta_1,\ldots,\beta_i)$ be an integer solution of (3.3) and $\mathrm{Sum}(A)=\sum_{t=1}^{i}(\alpha_t+\beta_t)$. For any integers h and p with $1\le h\le i$, let $A^{'}=(\alpha^{'}_1,\ldots,\alpha^{'}_i,\beta^{'}_1,\ldots,\beta^{'}_i)$, where $\alpha^{'}_h=\alpha_h+p(n_2+1)$, $\beta^{'}_h=\beta_h-pn_2$ and $\alpha^{'}_t=\alpha_t$, $\beta^{'}_t=\beta_t$ when $t\ne h$. We can see that $A^{'}$ is also an integer solution of (3.3) and

$$\mathrm{Sum}(A^{'})=\sum_{t=1}^{i}(\alpha^{'}_t+\beta^{'}_t)=\sum_{t=1}^{i}(\alpha_t+\beta_t)+p(n_2+1)-pn_2=\mathrm{Sum}(A)+p.$$

Therefore, suppose that (α,β) and $(\alpha^{'},\beta^{'})$ are the non-negative solutions of $\alpha n_2+\beta(n_2+1)=n_1+1$ and $\alpha^{'}n_2+\beta^{'}(n_2+1)=n_1$ with minimum integers α and $\alpha^{'}$, respectively, we can see that

$$A=(\alpha,\ldots,\alpha,\alpha^{'},\ldots,\alpha^{'},\beta,\ldots,\beta,\beta^{'},\ldots,\beta^{'})$$

is a non-negative integer solution of (3.3) and $\mathrm{Sum}(A)$ is minimum among all non-negative integer solutions of (3.3). Similarly, we can find another non-negative integer solution $A^{'}$ such that $\mathrm{Sum}(A^{'})$ is maximum. So, if $\mathrm{Sum}(A)\le j\le\mathrm{Sum}(A^{'})$, (3.3) has a non-negative integer solution satisfying (3.5). Hence, we obtain the following algorithm and result.

Algorithm 1:
Input: Three integers i, j and k with $1\le i<j<k$ and $k>2j$;
Output: Whether exists a permutation from $O(\alpha,k;i)$ to $O(\alpha,k;j)$;

1. set $n_1:=\lfloor\frac{k-1}{i}\rfloor$, $m_1:=(k-1)\%i+1$, $n_2:=\lfloor\frac{k-1}{j}\rfloor$, $m_2:=(k-1)\%j+1$;
2. set $\alpha:=-(n_1+1)$, $\beta:=n_1+1$ and $\alpha^{'}:=\beta^{'}:=0$;
3. set $q:=\lceil\frac{-\alpha}{n_2+1}\rceil$ and $\alpha:=\alpha+q(n_2+1)$, $\beta:=\beta-qn_2$;
   if $\beta<0$, return **False**; End. //whether (3.3) has a non-negative solution
   else, **Goto 4**;

4. set $\alpha^{'} := \alpha + 1, \beta^{'} := \beta - 1$;
   if $\alpha^{'} = n_2 + 1$, set $\alpha^{'} := 0$ and $\beta^{'} := \beta^{'} + n_2$;
   if $\beta^{'} < 0$, return **False**; End. //whether (3.3) has a non-negative solution
   else **Goto 5**;

5. set $j^{'} := m_1(\alpha + \beta) + (i - m_1)(\alpha^{'} + \beta^{'})$;
   if $j^{'} > j$, return **False**; End. //whether Sum$(A) \leq j$
   else, set $q_1 := \lfloor \frac{n_1+1-\alpha n_2}{n_2(n_2+1)} \rfloor, q_2 := \lfloor \frac{n_1-\alpha n_2}{n_2(n_2+1)} \rfloor$ and $j^{'} := j^{'} + m_1 q_1 + (i - m_1) q_2$;
   if $j^{'} < j$, return **False**; End. //whether Sum$(A') \geq j$
   else return **True**; End.

Lemma 3.2 *Let α, k, i, j, D_1 and D_2 be defined as above. Suppose that $i + j < k$ and $2j < k$. There is a permutation from D_1 to D_2 if and only if the return value of Algorithm 1 is True. Furthermore, if the return value is True, D_1 can be embedded in D_2.*

4 An example

In this section, we consider a special case when $i = 1$. In this case, we can obtain the following lemma.

Lemma 4.1 *For any integers α, k, i and j with $1 = i < j < k$, $O(\alpha, k; 1)$ can be embedded in $O(\alpha, k; j)$.*

Proof Firstly, we construct graph G as Sect. 3. Since $i = 1$, we can see that $G \cong P_k$. If $j = k - 1$, by Lemma. 2.1, $O(\alpha, k; 1)$ can be embedded in $O(\alpha, k; j)$. If $\lceil k/2 \rceil \leq j < k - 1$, then $i + j < k$ and $2j \geq k$. Since the cardinality of maximum matching of G is $\lfloor k/2 \rfloor \geq k - j$, $O(\alpha, k; 1)$ can be embedded in $O(\alpha, k; j)$. Otherwise, let $k = n_2 j + m_2$ and $G^{'} = m_2 P_{n_2+1} \bigcup (j - m_2) P_{n_2}$ as Sect. 3. It is easy to see that $G^{'}$ is a spanning subgraph of G as Fig. 1. Hence, $O(\alpha, k; 1)$ can be embedded in $O(\alpha, k; j)$. □

Furthermore, we can obtain a permutation Π efficiently. If $j = k-1$, we can obtain it by Lemma 2.1. In other cases, let $k = n_2 j + m_2$ and construct graphs $G \cong P_k$ and $G^{'} \cong m_2 P_{n_2+1} \bigcup (j - m_2) P_{n_2}$ (if $2j \geq k$, we see a matching with cardinality $k - j$ as $(k - j) P_2 \bigcup (2j - k) P_1$). It is easy to see that we can get a permutation Π from following algorithm.

Algorithm 2:
Input: Two integers j and k with $1 < j < k$;
Output: A permutation $\Pi = \binom{1\ 2\ \cdots\ k}{\pi_1\ \pi_2\ \cdots\ \pi_k}$ from $O(\alpha, k; 1)$ to $O(\alpha, k; j)$ for any integer α;

1 set $t := 0$ and $q := 1$;

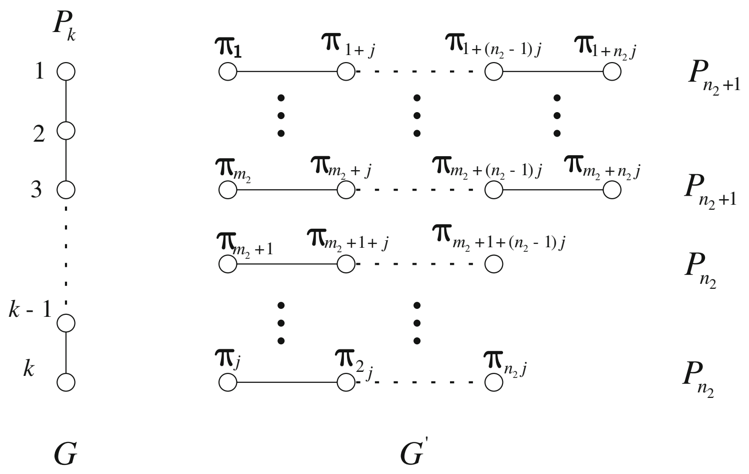

Fig. 1 $G \cong P_k$ and $G' \cong m_2 P_{n_2+1} \bigcup (j - m_2) P_{n_2}$

2 if $q > j$, then all elements of $\{\pi_1, \ldots, \pi_k\}$ are valued; End.
else, set $p := q$, $t := t + 1$ and $\pi_p := t$;

3 set $p := p + j$;
if $p > k$, set $q := q + 1$, **Goto 2**;
else, set $t := t + 1$, $\pi_p := t$, **Goto 3**;

It is easy to see that the time complexity of Algorithm 2 is $O(k)$. Furthermore, from Lemmas 2.1,3.1 and 3.2 and Algorithm 1, if there are permutations from $O(\alpha, k; i)$ to $O(\alpha, k; j)$, we can also construct such a permutation in $O(k)$ time.

A directed graph H of order n is *pancyclic* if it has cycles of all length $3, 4, \ldots, n$. Every directed de Bruijn graph $B(\alpha, k)$ is pancyclic (cf. Refs. [5] and [1], p. 308). By Lemma 4.1, every directed de Bruijn graph can be embedded in alphabet overlap digraph with the same parameters α and k. So we can obtain the following result.

Corollary 4.2 *Every alphabet overlap digraph is pancyclic.*

Lemma 4.3 *Let α, k, i be defined as above and d a common divisor of k and i. Then* $O(\alpha, k; i) \cong O(\alpha^d, k/d; i/d)$.

Proof Let $k' = k/d$ and $i' = i/d$. For convenience, we set $D_1 = O(\alpha, k; i)$ and $D_2 = O(\alpha^d, k'; i')$ and replace alphabet set $\{1, \ldots, \alpha\}$ by $\{0, \ldots, \alpha - 1\}$. Firstly, we give a map f from $V(D_1)$ to $V(D_2)$ as follows. For any vertex $v = (v_1, \ldots, v_k)$ of D_1, let $f(v) = (f_1, \ldots, f_{k'})$ be a vertex of D_2, where $f_t = v_{(t-1)d+1}\alpha^{(d-1)} + \cdots + v_{td}\alpha^0$, for every $t \in \{1, \ldots, k'\}$. This implies that $(v_{(t-1)d+1}, \ldots, v_{td})$ is the representation of f_t by α-nary numeral system with d-digit. Hence, we can see that f is a bijection.

Furthermore, let u and v be two vertices of D_1 such that (v, u) is an arc of D_1. Suppose that $u = (u_1, \ldots, u_k)$ and $f(u) = (f_1^{'}, \ldots, f_{k'}^{'})$. We have that

$$
\begin{aligned}
&(v, u) \text{ is an arc of } D_1;\\
\cong\ & \text{for any } t \geq i+1,\ v_t = u_{t-i};\\
\cong\ & \text{for any } t^{'} \in \{i^{'}+1, \ldots, k^{'}\},\ (v_{(t'-1)d+1}, \ldots, v_{t'd})\\
=\ & (u_{(t'-i'-1)d+1}, \ldots, u_{(t'-i')d});\\
\cong\ & \text{for any } t^{'} \in \{i^{'}+1, \ldots, k^{'}\},\ f_{t'} = f_{t'-i'}^{'};\\
\cong\ & (f(v), f(u)) \text{is an arc of } D_2.
\end{aligned}
$$

Hence, $D_1 \cong D_2$. □

Now, we consider a special case when i is a common divisor of k and j. Let $k = k^{'}i$ and $j = j^{'}i$. By Lemma 4.3, we have $O(\alpha, k; i) \cong O(\alpha^i, k^{'}; 1)$ and $O(\alpha, k; j) \cong O(\alpha^i, k^{'}; j^{'})$. By Lemma 4.1, $O(\alpha^i, k^{'}; 1)$ can be embedded in $O(\alpha^i, k^{'}; j^{'})$. So we obtain the following result.

Corollary 4.4 *Let α, k, i and j be defined as above. If i is a common divisor of k and j, $O(\alpha, k; i)$ can be embedded in $O(\alpha, k; j)$.*

5 Conclusion

In this paper, we consider the problem: for given integers α, i, j and k with $1 \leq i < j < k$, whether $O(\alpha, k; i)$ can be embedded in $O(\alpha, k; j)$. In order to resolve the problem, we pose another problem: for given integers α, i, j and k with $1 \leq i < j < k$, whether there exists a permutation Π from $O(\alpha, k; i)$ to $O(\alpha, k; j)$. We solve the second problem completely. The main result of this paper is obtained by Lemmas 2.1, 3.1 and 3.2 as follows.

Theorem 5.1 *For given integers α, i, j and k with $1 \leq i < j < k$, there exists a permutation Π from $O(\alpha, k; i)$ to $O(\alpha, k; j)$ if and only if*

(1) $m_1\lfloor\frac{n_1+1}{2}\rfloor + (i - m_1)\lfloor\frac{n_1}{2}\rfloor \geq k - j$ *(where n_1 and m_1 are two integers with $k = n_1 i + m_1$ and $1 \leq m_1 \leq i$), when $2j \geq k$;*

(2) *the return value of Algorithm 1 is True, when $2j < k$.*

Furthermore, if there exist permutations from $O(\alpha, k; i)$ to $O(\alpha, k; j)$, $O(\alpha, k; i)$ can be embedded in $O(\alpha, k; j)$ and we can construct such a permutation in $O(k)$ time.

Remark 1 If $i + j \geq k$, we have that $2j \geq k$ and $m_1\lfloor\frac{n_1+1}{2}\rfloor + (i - m_1)\lfloor\frac{n_1}{2}\rfloor \geq k - j$. Hence, Lemmas 2.1 and 3.1 are combined into Theorem 5.1.

Note that even if there is no a permutation from $O(\alpha, k; i)$ to $O(\alpha, k; j)$, $O(\alpha, k; i)$ may be embedded in $O(\alpha, k; j)$. For example, there is no a permutation from $O(2, 6; 2)$ to $O(2, 6; 3)$, but $O(2, 6; 2)$ can be embedded in $O(2, 6; 3)$. Hence, we only solve partly the first problem. The future work is to find some new methods to resolve the first problem completely.

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